COMPARATIVE STUDY OF DUAL BOX COVERING ALGORITHMS FOR FRACTAL GRAPHS DIMENSION COMPUTATION

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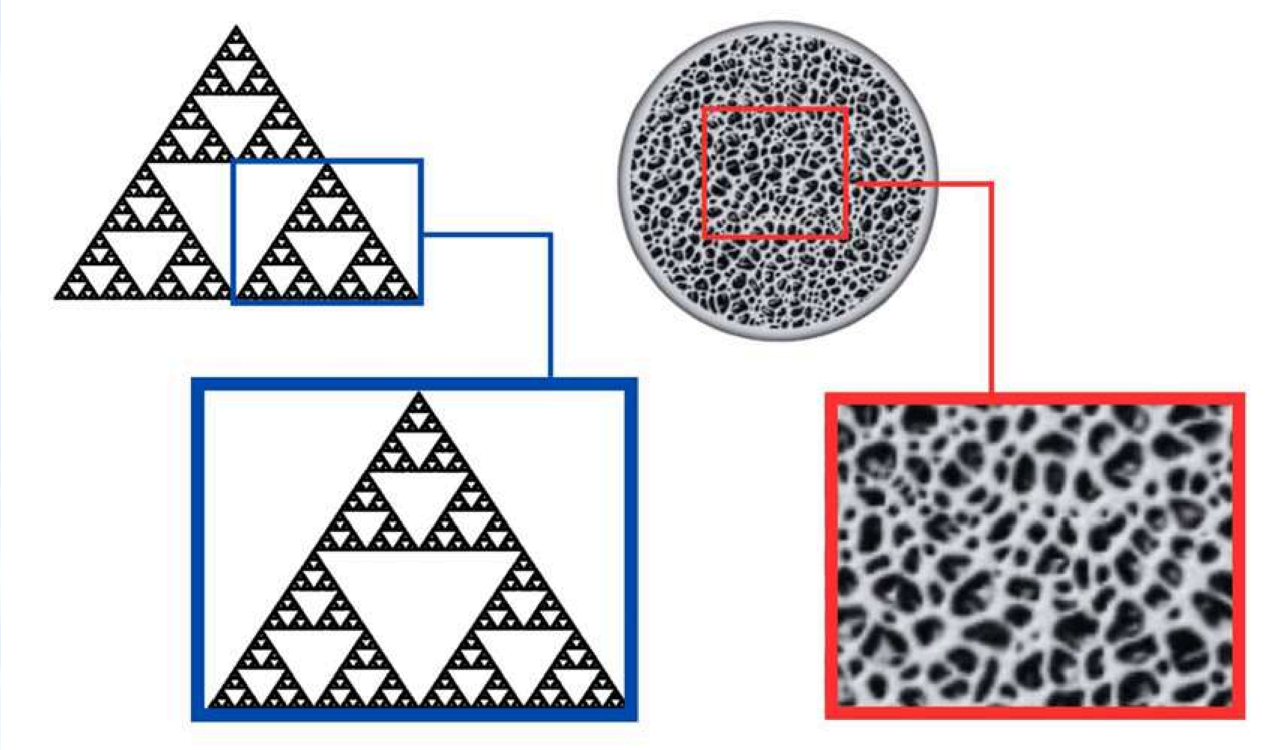
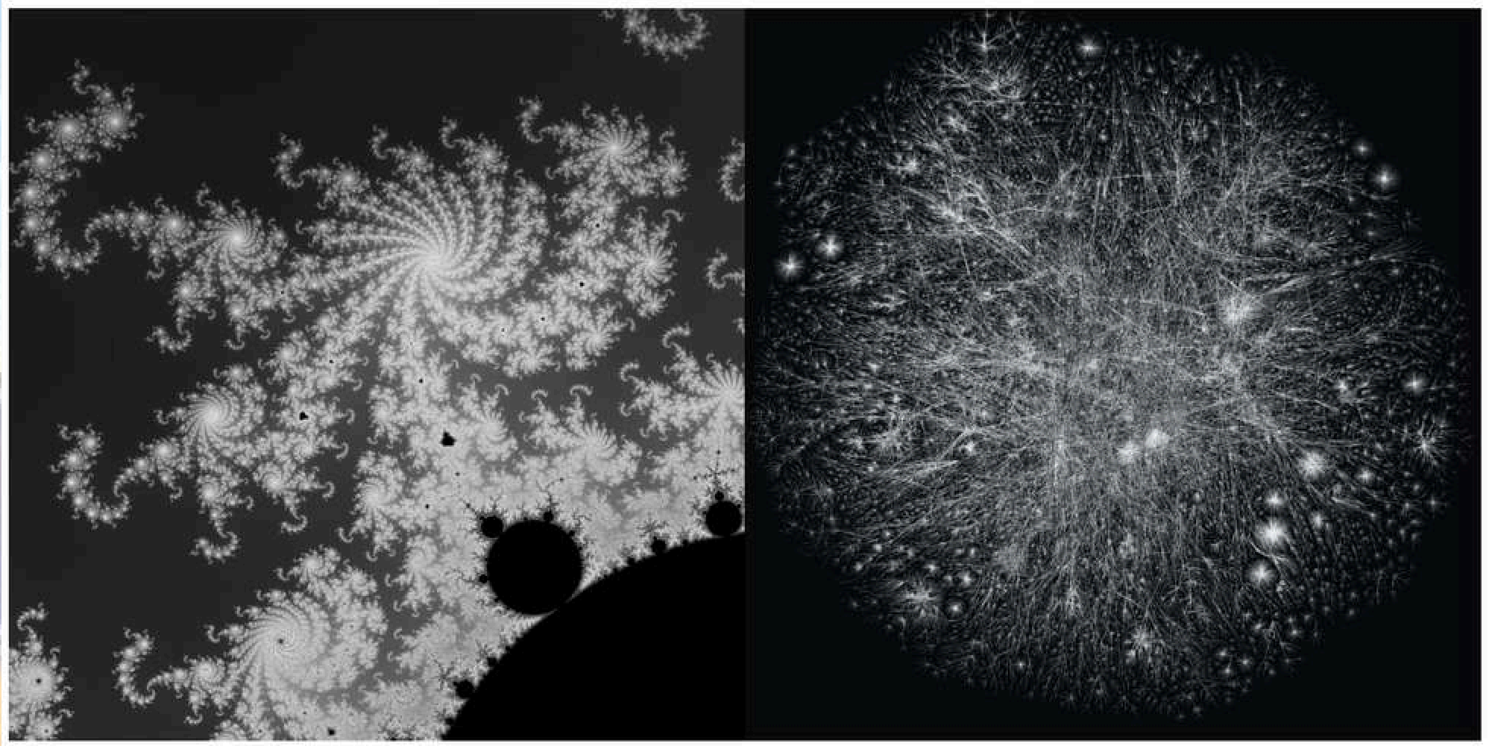
NSTC | International Internship Pilot Program



Self-similarity



Scale-free complexity

$$H_r^s(X) = \inf_{\operatorname{diam}(A_i)<r} \left\{ \sum_{i=1}^{+\infty} \operatorname{diam}(A_i)^s \mid X \subseteq \bigcup_{i=1}^{+\infty} A_i \right\}$$

$$H^s(X) = \lim_{r \to 0^+} H_r^s(X)$$

$$D_H(X) = \inf \left\{ s, H^s(X) = 0 \right\} = \sup \left\{ s, H^s(X) = \infty \right\}$$

$$D_{h'} = \frac{ln(N)}{ln(\frac{1}{r})}$$

Fractal dimension

If D_{box} limit doesn't exist, we usually denote:

$$D_{\text{box}} = \lim_{\varepsilon \to 0^+} \frac{\ln(N(\varepsilon))}{\ln(\frac{1}{\varepsilon})} \qquad D_{\overline{\text{box}}} = \overline{\lim_{\varepsilon \to 0^+}} \frac{\ln(N(\varepsilon))}{\ln(\frac{1}{\varepsilon})} \quad \text{and} \quad D_{\underline{\text{box}}} = \underline{\lim_{\varepsilon \to 0^+}} \frac{\ln(N(\varepsilon))}{\ln(\frac{1}{\varepsilon})}$$

1. number of squares in a ε-grid covering the set
2. smallest number of balls of radius ε covering the set
3. smallest number of cubes of side ε covering the set
4. smallest number of subsets of diameter ε covering the set
5. largest number of disjoint ε-balls centered on the set

Box-counting fractal dimension

$$D_H \leq D_{\text{box}}$$

In case of exact self-similarity, D_H and D_{box} are conjectured to be equal

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2. Problems

Box Cover problems

$$\mathcal{B}(u,\varepsilon) = \{v \in V \mid dist(u,v) \leq \varepsilon\}$$

$$\mathcal{D}(G,\varepsilon) = \{D \subseteq V \mid (\mathcal{B}(d,\varepsilon))_{d\in D} \text{ are pairwise disjoint. }\}$$

$$\mathcal{C}(G,\varepsilon) = \{C \subseteq V \mid V = \textstyle\bigcup_{c\in C} \mathcal{B}(c,\varepsilon)\}$$

Complete Box Cover decision problem:

- Input: $G = (V,E)$, $\varepsilon \in \mathbb{R}$, $k \in \mathbb{N}$
- Output: Is there a vertex set $C \in \mathcal{C}(G,\varepsilon)$ such that $|C| = k$?

Complete Box Cover minimization problem:

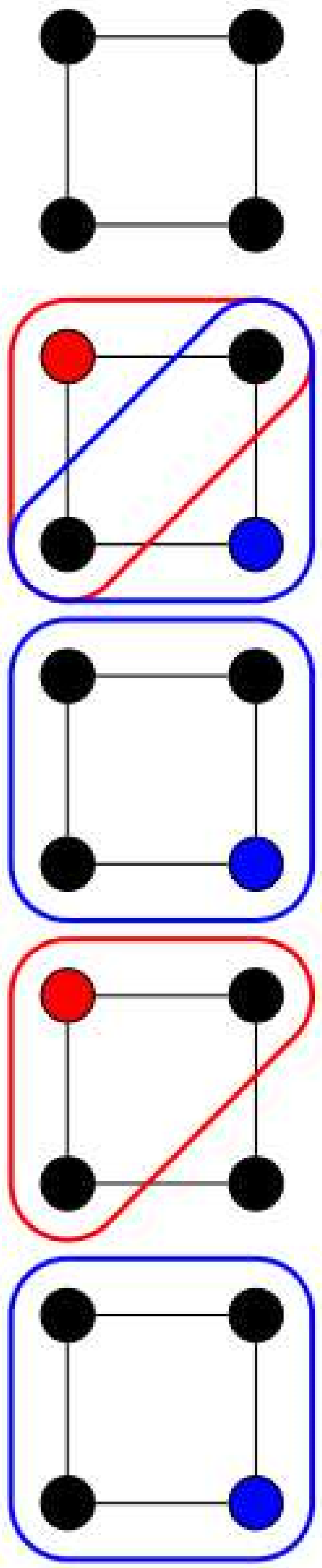
- Input: $G = (V,E)$, $\varepsilon \in \mathbb{R}$
- Output: What is the size of the smaller set $C \in \mathcal{C}(G,\varepsilon)$?

Disjoint Box Cover decision problem:

- Input: $G = (V,E)$, $\varepsilon \in \mathbb{R}$, $k \in \mathbb{N}$
- Output: Is there a vertex set $D \in \mathcal{D}(G,\varepsilon)$ such that $|D| = k$?

Disjoint Box Cover maximization problem:

- Input: $G = (V,E)$, $\varepsilon \in \mathbb{R}$
- Output: What is the size of the greater set $D \in \mathcal{D}(G,\varepsilon)$?



YZU - CSDBCAFGDC : 2. Problems

Box Cover order relationship

$$\overline{\mathcal{D}}(G,\varepsilon) = \{D \in \mathcal{D}(G,\varepsilon) \mid \mathrm{DBC}(G,\varepsilon) = |D|\}$$

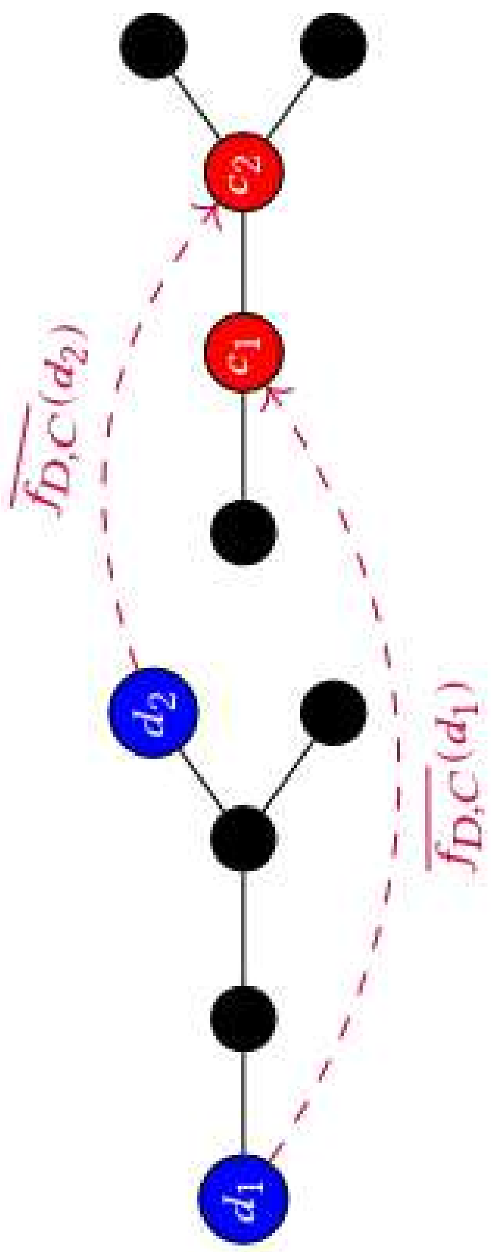
$$\overline{C}(G,\varepsilon) = \{C \in C(G,\varepsilon) \mid \mathrm{CBC}(G,\varepsilon) = |C|\}$$

THEOREM 2.1. $\forall\ (D,C) \in \mathcal{D}(G,\varepsilon) \times C(G,\varepsilon),\ |D| \leq |C|$

$f_{D,C} : D \to \mathcal{P}(C)$ defined as follows:

$\forall\ d \in D,\ f_{D,C}(d) = \{c \in C \mid d \in \mathcal{B}(c,\varepsilon)\}$

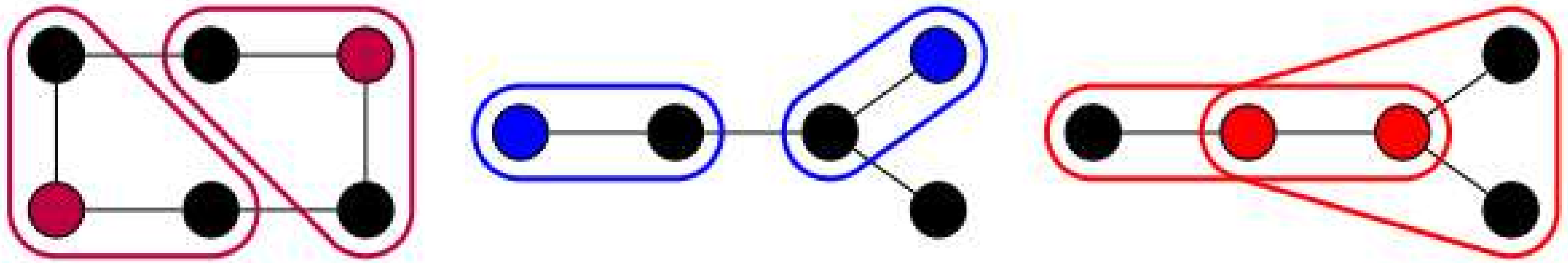
COROLLARY 2.3.1. $DBC(\cdot,\cdot) \leq CBC(\cdot,\cdot)$



THEOREM 2.4. $\forall\ G = (V,E),\ \varepsilon \in \mathbb{R},\ (DBC(G,\varepsilon) = CBC(G,\varepsilon))$
$\Leftrightarrow \forall\ (D,C) \in \overline{\mathcal{D}}(G,\varepsilon) \times \overline{C}(G,\varepsilon),\ f_{D,C}$ *is bijective.*

Disjoint Complete Box Cover decision problem:

- Input: $G = (V,E),\ \varepsilon \in \mathbb{R}$
- Output: Is there a vertex set $B \in \mathcal{D}(G,\varepsilon) \cap C(G,\varepsilon)$?



Box Cover characterizations

Box centers b_1 and b_2 are called *ε-consistent* if:

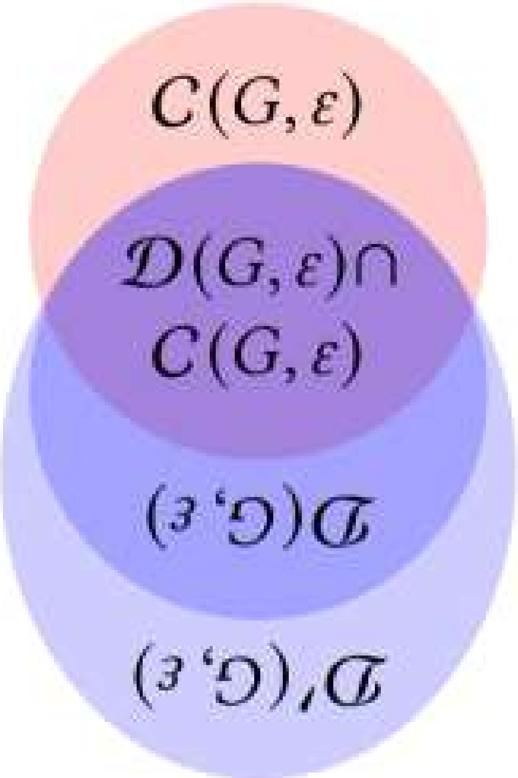
$$\forall\ (u_1, u_2) \in \mathcal{B}(b_1, \varepsilon) \times \mathcal{B}(b_2, \varepsilon) \cap E,$$

$$w(u_1, u_2) + \max(\mathrm{dist}(u_1, b_1),\ \mathrm{dist}(u_2, b_2)) > \varepsilon$$

A box cover $B \subseteq V$ is *ε-consistent* if:

$$\forall\ b_1 \neq b_2 \in B,\ b_1 \text{ and } b_2 \text{ are } \varepsilon\text{-consistent}$$

we call $\mathcal{D}'(G, \varepsilon)$ the set of ε-consistent box covers



Theorem 2.6. $\mathcal{D}(G, \varepsilon) \subset \mathcal{D}'(G, \varepsilon)$

Theorem 2.7. $\mathcal{C}(G, \varepsilon) \cap \mathcal{D}(G, \varepsilon) = \mathcal{C}(G, \varepsilon) \cap \mathcal{D}'(G, \varepsilon)$

$$\mathcal{U}(D, \varepsilon) = \{v \in V \mid \nexists\, d \in D \mid v \in \mathcal{B}(d, \varepsilon)\}$$

$$\mathcal{O}(C, \varepsilon) = \{v \in V \mid \exists\, c \neq c' \in C \mid v \in \mathcal{B}(c, \varepsilon) \cap \mathcal{B}(c', \varepsilon)\}$$

Theorem 2.8. *If $DBC(G, \varepsilon) = CDC(G, \varepsilon)$, uncovered / overcovered vertices of optimal disjoint / complete box covers cannot be vertices of optimal complete / disjoint box covers.*

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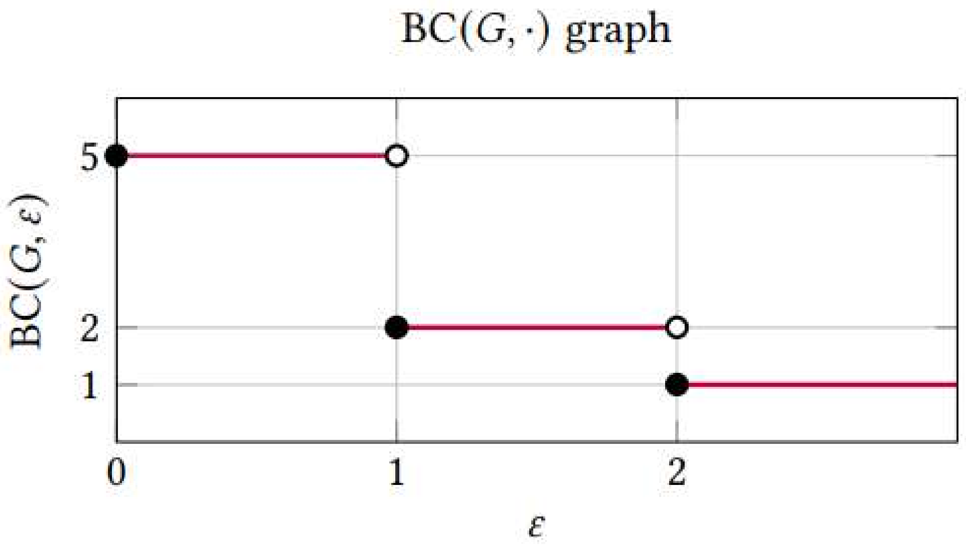
Box Cover dependency on epsilon

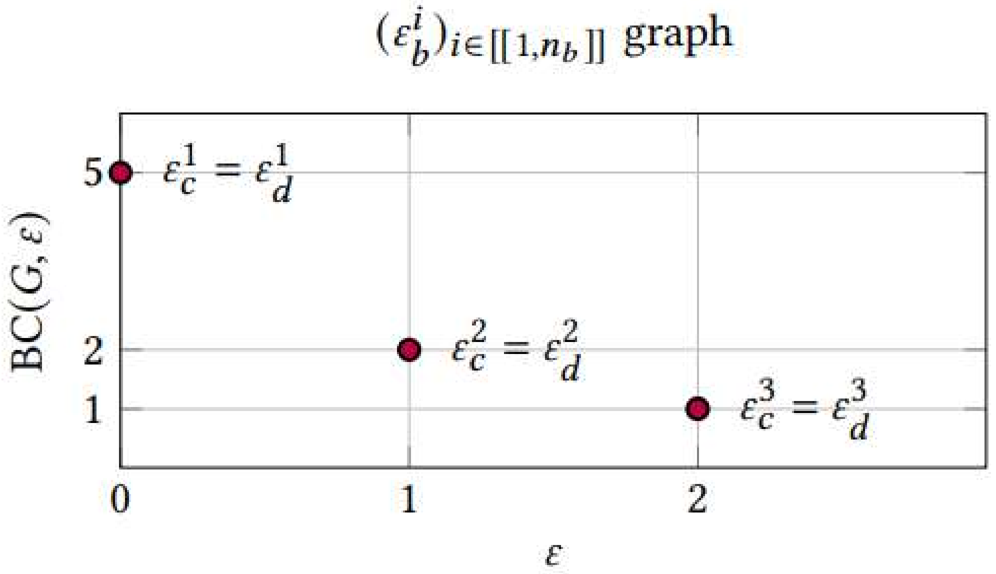
THEOREM 2.11. *$CBC(G, \cdot)$ is a decreasing function.*

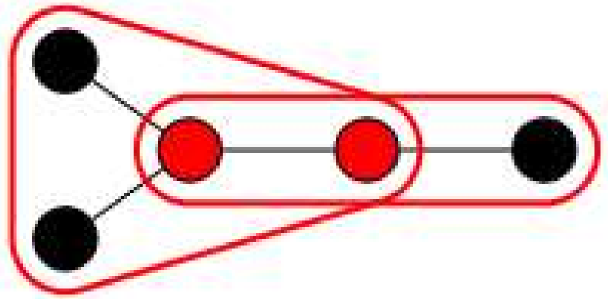
THEOREM 2.13. *$DBC(G, \cdot)$ is a decreasing function.*

THEOREM 2.15. *$CBC(G, \cdot)$ is a left-continuous step function.*

THEOREM 2.19. *$DBC(G, \cdot)$ id a left-continuous step function.*







THEOREM 2.26. $\forall \varepsilon > 0, CBC(\cdot, 2\varepsilon) \leq DBC(\cdot, \varepsilon)$

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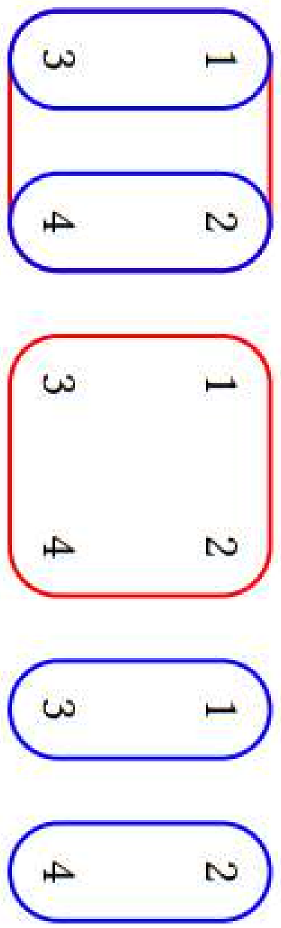
3. Complexity

NP-complexity of box cover problems

THEOREM 3.1. *CBC-DEC, DBC-DEC* $\in \mathcal{NP}$

THEOREM 3.2. *CBC-DEC and DBC-DEC are* $\mathcal{NP}$*-complete.*

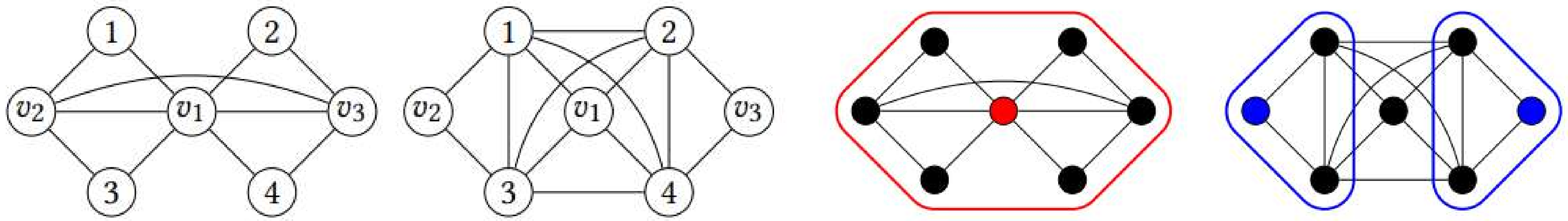


Set Cover decision problem:

- Input: $U = [\![1, n]\!]$, $S \subseteq \mathcal{P}(U)$ such that $U = \bigcup_{s \in S} s$, $k \leq |S|$
- Output: Is there a subset set $S' \subseteq S$ such that $|S'| = k$ and $U = \bigcup_{s \in S} s$?

Set Packing decision problem:

- Input: $U = [\![1, n]\!]$, $S \subseteq \mathcal{P}(U)$, $k \leq |S|$
- Output: Is there a subset set $S' \subseteq S$ such that $|S'| = k$ and $(s)_{s \in S'}$ are pairwise disjoint?



THEOREM 3.9. *CBC and DBC are* $\mathcal{NP}$*-hard.*

Box Cover Integer Linear Programs

$$\forall\, i \in [[1,n]],\ b_i = \begin{cases} 1 & \text{if vertex } v_i \text{ is the center of a box} \\ 0 & \text{otherwise} \end{cases}$$

$$\forall\, i,j \in [[1,n]],\ \varepsilon_{i,j} = \begin{cases} 1 & \text{if } \operatorname{dist}(v_i, v_j) \leq \varepsilon \\ 0 & \text{otherwise} \end{cases}$$

$$\begin{array}{lll} \text{minimize} & \sum_{i=1}^{n} c_i & \\ \text{subject to} & \sum_{i=1}^{n} \varepsilon_{i,j} \times c_i \geq 1 & \forall\, j \in [[1,n]] \\ & c_i \in \{0,1\} & \forall\, i \in [[1,n]] \end{array} \qquad \begin{array}{ll} \text{minimize} & \mathbb{1}^T \times C \\ \text{subject to} & E \times C \geq \mathbb{1} \\ & C \in \mathcal{M}_{n,1}(\{0,1\}) \end{array}$$

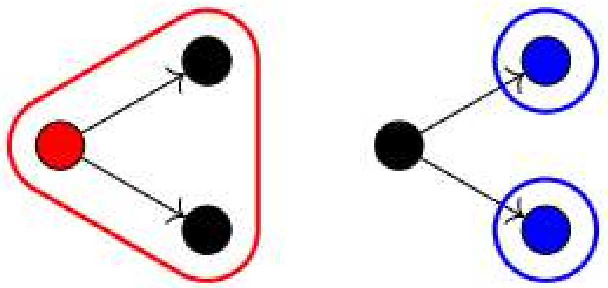
$$\begin{array}{lll} \text{maximize} & \sum_{i=1}^{n} d_i & \\ \text{subject to} & \sum_{i=1}^{n} \varepsilon_{i,j} \times d_i \leq 1 & \forall\, j \in [[1,n]] \\ & d_i \in \{0,1\} & \forall\, i \in [[1,n]] \end{array} \qquad \begin{array}{ll} \text{maximize} & \mathbb{1}^T \times D \\ \text{subject to} & E \times D \leq \mathbb{1} \\ & D \in \mathcal{M}_{n,1}(\{0,1\}) \end{array}$$

THEOREM 4.1. $E = \max(0, 1 - \max(0, \frac{D}{\varepsilon} - 1))$

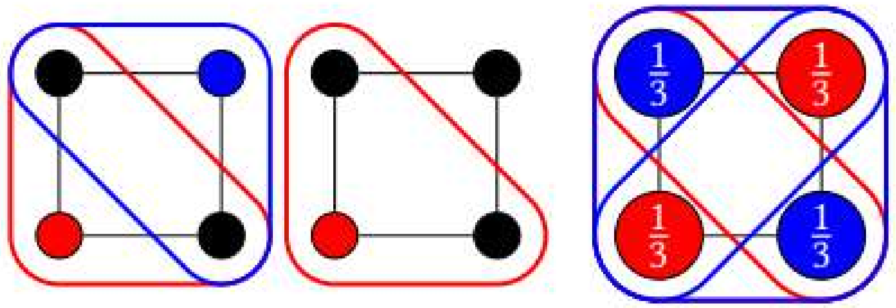
THEOREM 4.2. *CBC and DBC linear programs are duals.*

Box Cover Problems Variants

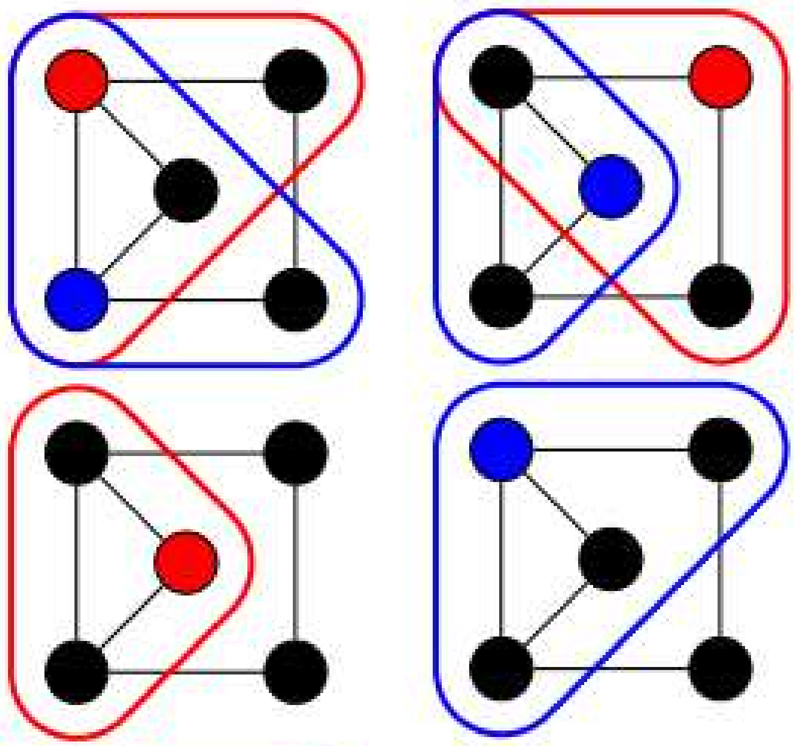
Oriented box cover



Fractional box cover



Weighted box cover



Infinite graphs box covers

$$
\begin{array}{ll}
\text{minimize} & \sum_{i=1}^{+\infty} c_i \\
\text{subject to} & \sum_{i=1}^{+\infty} \varepsilon_{i,j} \times c_i \geq 1 \quad \forall\, j \geq 1 \\
 & c_i \in \{0, 1\} \quad \forall\, i \geq 1
\end{array}
$$

$$
\begin{array}{ll}
\text{maximize} & \sum_{i=1}^{+\infty} d_i \\
\text{subject to} & \sum_{i=1}^{+\infty} \varepsilon_{i,j} \times d_i \leq 1 \quad \forall\, j \geq 1 \\
 & d_i \in \{0, 1\} \quad \forall\, i \geq 1
\end{array}
$$

Box Cover Greedy Algorithms

$$\forall\, u, v \in V,\ u \preceq_V v \Rightarrow \deg_V(u) \leq \deg_V(v)$$

CBC heuristic consists in picking as centers of complete box covers the maximum vertices according to $\preceq_V$.

DBC heuristic consists in picking as center of disjoint box covers the minimum vertices according to $\preceq_V$.

Algorithm 7 Greedy approximation algorithm for Complete Box Cover

Require: $G = (V, E)$, $\varepsilon \in \mathbb{R}$
Ensure: $V = \bigcup_{c \in C} \mathcal{B}(c, \varepsilon)$
  $P \leftarrow V, C \leftarrow \emptyset$
  while $|P| \neq 0$ **do**
    Choose c in P according to CBC Heuristic
    $P \leftarrow P \setminus \mathcal{B}(c, \varepsilon)$
    $C \leftarrow C \cup \{c\}$
  end while

Algorithm 8 Greedy algorithm for Disjoint Box Cover

Require: $G = (V, E)$, $\varepsilon \in \mathbb{R}$
Ensure: $(\mathcal{B}(d, \varepsilon))_{d \in D}$ are pairwise disjoint.
  $P \leftarrow V, D \leftarrow \emptyset$
  while $|P| \neq 0$ **do**
    Choose d in P according to DBC Heuristic
    $P \leftarrow P \setminus \mathcal{B}(d, \varepsilon)$
    $D \leftarrow D \cup \{d\}$
    $P \leftarrow P \setminus \{p \in P \mid \mathcal{B}(d, \varepsilon) \cap \mathcal{B}(p, \varepsilon) \neq \emptyset\}$ ▷ i.e we remove from potential centers the vertices whose boxes would intersect with the box added to D
  end while

THEOREM 6.1. *Algorithm 7 is not an approximation for CBC.*

THEOREM 6.2. *Algorithm 8 is not an approximation for DBC.*

YZU - CSDBCAFGDC :
6. Algorithms

Box Cover Conversion Greedy Algorithms

- 1. We choose as much centers in the argument box cover as possible thanks to BC heuristics
- 2. If the box cover obtained can possibly (or need to) be improved, we run the precedent greedy algorithms on the remaining vertices (not belonging to the original cover)

Algorithm 9 Complete Box Cover to Disjoint Box Cover provider

Require: $G = (V, E), \varepsilon \in \mathbb{R}$ | $\mathrm{DBC}(G, \varepsilon) = \mathrm{CBC}(G, \varepsilon)$, $C \in \mathcal{C}(G, \varepsilon)$
Ensure: $(\mathcal{B}(d, \varepsilon))_{d \in D}$ are pairwise disjoint.
$D \leftarrow \emptyset, P \leftarrow V \setminus \mathcal{O}(C)$
while $|P| \neq 0$ **do**
Choose d in P according to DBC Heuristic
Choose c in $f_{D,P}(d)$ according to DBC Heuristic
if $\mathcal{B}(d, \varepsilon) \cap \biguplus_{d' \in D} \mathcal{B}(d', \varepsilon) = \emptyset$ **then**
$P \leftarrow P \setminus \mathcal{B}(c, \varepsilon)$
$D \leftarrow D \cup \{d\}$
else
$P \leftarrow P \setminus \{d\}$
end if
end while
$V' \leftarrow V \setminus \biguplus_{d \in D} \mathcal{B}(d, \varepsilon)$
if $V' \neq \emptyset$ **then**
$D' \leftarrow$ AlgoDBC $(G[V'], \varepsilon)$
$D \leftarrow D \cup D'$
end if

Algorithm 10 Disjoint Box Cover to Complete Box Cover provider

Require: $G = (V, E), \varepsilon \in \mathbb{R}, D \in \mathcal{D}(G, \varepsilon)$
Ensure: $V = \bigcup_{c \in C} \mathcal{B}(c, \varepsilon)$
$C \leftarrow \emptyset, P \leftarrow V \setminus \mathcal{U}(D)$
while $|P| \neq 0$ **do**
Choose c in P according to CBC Heuristic
$d \leftarrow f_{C,P}(c)$
$P \leftarrow P \setminus \mathcal{B}(d, \varepsilon)$
$C \leftarrow C \cup \{c\}$
end while
$V' \leftarrow V \setminus \bigcup_{c \in C} \mathcal{B}(c, \varepsilon)$
if $V' \neq \emptyset$ **then**
$C' \leftarrow$ AlgoCBC $(G[V'], \varepsilon)$
$C \leftarrow C \cup C'$
end if

THEOREM 6.3. *Algorithm 9 is not an approximation for CBC to DBC conversion with DCBC-DEC true and optimal box covers provided.*

THEOREM 6.4. *Algorithm 10 is not an approximation for DBC to CBC conversion with DCBC-DEC true and optimal box covers provided.*

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YZU - CSDBCAFGDC : 7. Fractals

Fractal Graphs Formalization

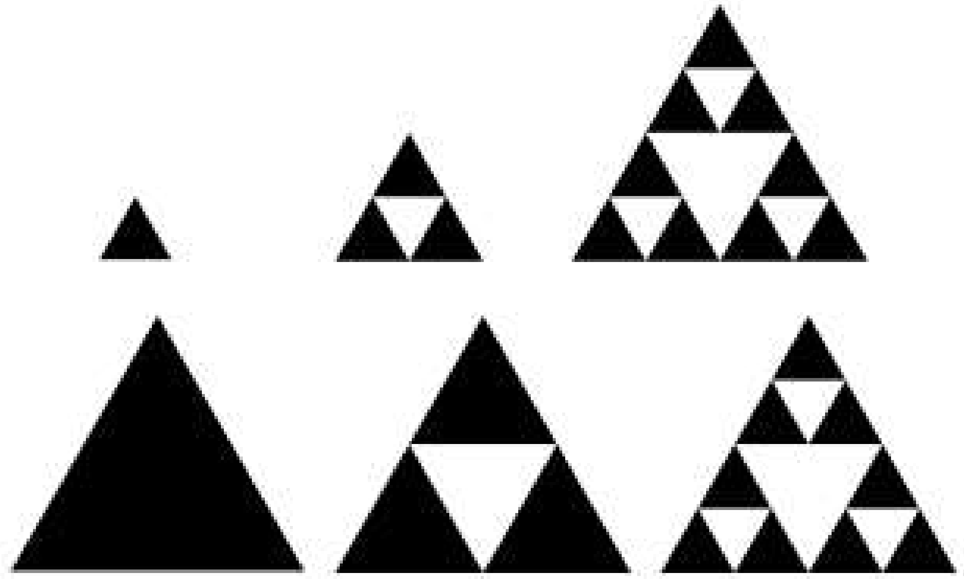
$$F = \lim_{n \to +\infty} F_n$$

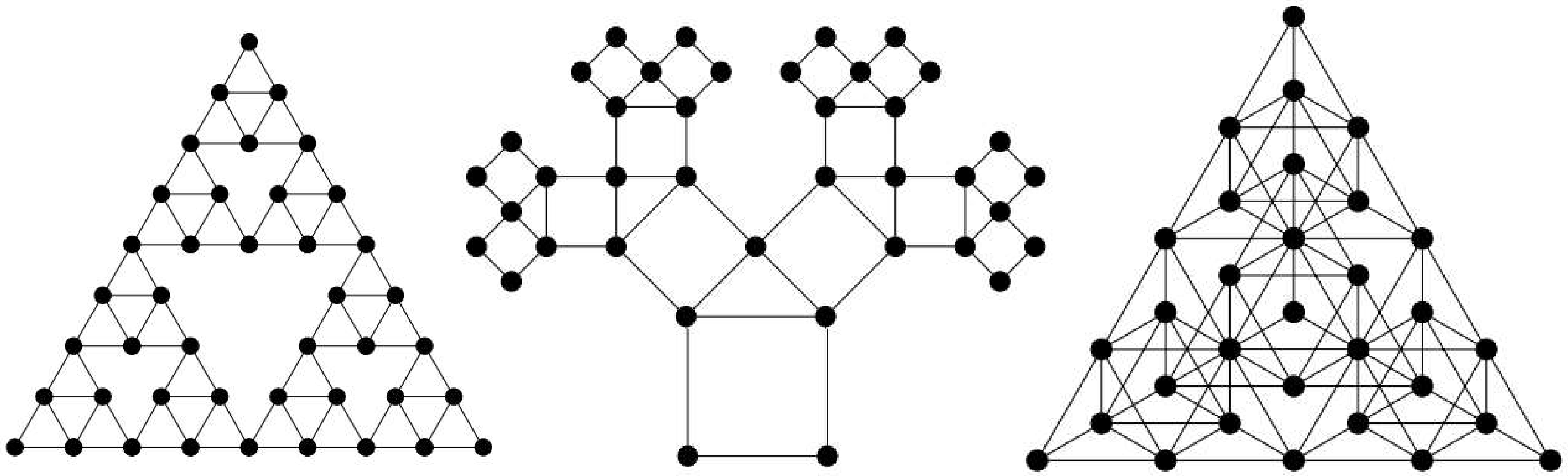
Proposition 7.1. $\forall F \in \mathcal{F},\ n \geq 1,\ u, v \in V_{F_n},$

$$dist_{V_{\overline{F_n}}}(u, v) = dist_{V_{\overline{F_{n+1}}}}(u, v)$$

Proposition 7.2. $\forall F \in \mathcal{F},\ \exists K \in \mathbb{N} \mid \forall n \geq 1,$

$$\forall v \in V_{\overline{F_{n+1}}},\ \min_{u \in V_{\overline{F_n}}} dist(u, v) \leq K r^n$$





Theorem 7.5. $D_H(S) = \frac{\ln(3)}{\ln(2)}$ Theorem 7.6. $D_H(P) = 2$ Theorem 7.7. $D_H(T) = 2$

Box-counting dimension limits

$D(F)$	$D^c(F)$	$D^d(F)$
$D_{\text{box}}(F)$	$\lim_{\varepsilon\to 0^+} \frac{\ln(\text{CBC}(F,\varepsilon))}{\ln(\frac{1}{\varepsilon})}$	$\lim_{\varepsilon\to 0^+} \frac{\ln(\text{DBC}(F,\varepsilon))}{\ln(\frac{1}{\varepsilon})}$
$D_{\overline{\text{box}}}(F)$	$\overline{\lim}_{\varepsilon\to 0^+} \frac{\ln(\text{CBC}(F,\varepsilon))}{\ln(\frac{1}{\varepsilon})}$	$\overline{\lim}_{\varepsilon\to 0^+} \frac{\ln(\text{DBC}(F,\varepsilon))}{\ln(\frac{1}{\varepsilon})}$
$D_{\underline{\text{box}}}(F)$	$\underline{\lim}_{\varepsilon\to 0^+} \frac{\ln(\text{CBC}(F,\varepsilon))}{\ln(\frac{1}{\varepsilon})}$	$\underline{\lim}_{\varepsilon\to 0^+} \frac{\ln(\text{DBC}(F,\varepsilon))}{\ln(\frac{1}{\varepsilon})}$

THEOREM 7.1. $\forall\, F \in \mathcal{F}$,
If $D^c_{\text{box}}(F)$ or $D^d_{\text{box}}(F)$ does converge, then $D^c_{\text{box}}(F) = D^d_{\text{box}}(F)$.
Otherwise, $D^c_{\overline{\text{box}}}(F) = D^d_{\overline{\text{box}}}(F)$ and $D^c_{\underline{\text{box}}}(F) = D^d_{\underline{\text{box}}}(F)$.

$$c_n(\varepsilon) = \text{CBC}(\overline{F_n}, \varepsilon) \quad \text{and} \quad d_n(\varepsilon) = \text{DBC}(\overline{F_n}, \varepsilon)$$

THEOREM 7.4. $\forall\, F \in \mathcal{F}$, *depending on the convergence of $D_{box}(F)$:*

$$D_{box}(F) = \lim_{\varepsilon\to 0^+} \lim_{n\to+\infty} \frac{\ln(c_n(\varepsilon))}{\ln(\frac{1}{\varepsilon})} = \lim_{\varepsilon\to 0^+} \lim_{n\to+\infty} \frac{\ln(d_n(\varepsilon))}{\ln(\frac{1}{\varepsilon})}$$

$$D_{\overline{box}}(F) = \overline{\lim_{\varepsilon\to 0^+}} \lim_{n\to+\infty} \frac{\ln(c_n(\varepsilon))}{\ln(\frac{1}{\varepsilon})} = \overline{\lim_{\varepsilon\to 0^+}} \lim_{n\to+\infty} \frac{\ln(d_n(\varepsilon))}{\ln(\frac{1}{\varepsilon})}$$

$$D_{\underline{box}}(F) = \underline{\lim_{\varepsilon\to 0^+}} \lim_{n\to+\infty} \frac{\ln(c_n(\varepsilon))}{\ln(\frac{1}{\varepsilon})} = \underline{\lim_{\varepsilon\to 0^+}} \lim_{n\to+\infty} \frac{\ln(d_n(\varepsilon))}{\ln(\frac{1}{\varepsilon})}$$

YZU - CSDBCAFGDC :
8. Framing

Fractal Box Covers Framing

COROLLARY 8.6.1. $\forall\ n,\ \varepsilon \geq 1,$

$$\frac{3^{n-2}}{2\varepsilon^{\ln_2(3)}} \leq DBC(S_n, \varepsilon) \leq CBC(S_n, \varepsilon) \leq \frac{3^{n-2+\ln_2(3)}}{\varepsilon^{\ln_2(3)}}$$

COROLLARY 8.6.2. *$DBC(S_n, \varepsilon)$ and $CBC(S_n, \varepsilon)$ are* $\theta\left(\frac{3^n}{\varepsilon^{\ln_2(3)}}\right)$

COROLLARY 8.8.1. $\forall\ n,\ \varepsilon \geq 1,$

$$2^{n-3}\left(\frac{3+\sqrt{2}}{\varepsilon+1+\sqrt{2}}\right)^2 \leq DBC(P_n, \varepsilon) \leq CBC(P_n, \varepsilon) \leq 2^{n-1} \times 3\left(\frac{3+\sqrt{2}}{\varepsilon+1+\sqrt{2}}\right)^2$$

COROLLARY 8.8.2. *$DBC(P_n, \varepsilon)$ and $CBC(P_n, \varepsilon)$ are* $\theta\left(\frac{2^n}{(\varepsilon+1+\sqrt{2})^2}\right)$

COROLLARY 8.10.1. $\forall\ n,\ \varepsilon \geq 1,$

$$2\frac{4^{n-3}}{\varepsilon^2} \leq DBC(T_n, \varepsilon) \leq CBC(T_n, \varepsilon) \leq 49\frac{4^{n-3}}{\varepsilon^2}$$

COROLLARY 8.10.2. *$DBC(S_n, \varepsilon)$ and $CBC(S_n, \varepsilon)$ are* $\theta\left(\frac{4^n}{\varepsilon^2}\right)$

Fractal	Sierpinski Triangle	Pythagoras Tree	Sierpinski Tetrahedron
$f(F_n, \varepsilon)$	$3^n/\varepsilon^{\ln_2(3)}$	$2^n/(\varepsilon+1+\sqrt{2})^2$	$4^n/\varepsilon^2$
N	3	2	4
$D_H(F)$	$\ln_2(3)$	2	2

Schroeder's conjecture validity

THEOREM 8.11. $\forall F \in \mathcal{F}, \exists\, c \geq 0 \mid$ *DBC*(F_n, ε) *and CBC*(F_n, ε) *are* $\theta\left(\frac{N^n}{(\varepsilon+c)^{D_H(F)}}\right) \Leftrightarrow f(F_n, \varepsilon) = \frac{N^n}{(\varepsilon+c)^{D_H(F)}}$

COROLLARY 8.11.1. $\forall F \in \mathcal{F}, \exists\, c \geq 0 \mid$ *DBC*$(\overline{F_n}, \varepsilon)$ *and CBC*$(\overline{F_n}, \varepsilon)$ *are* $\theta\left((\varepsilon + cr^n)^{-D_H(F)}\right) \Leftrightarrow f\left(\overline{F_n}, \varepsilon\right) = (\varepsilon + cr^n)^{-D_H(F)}$

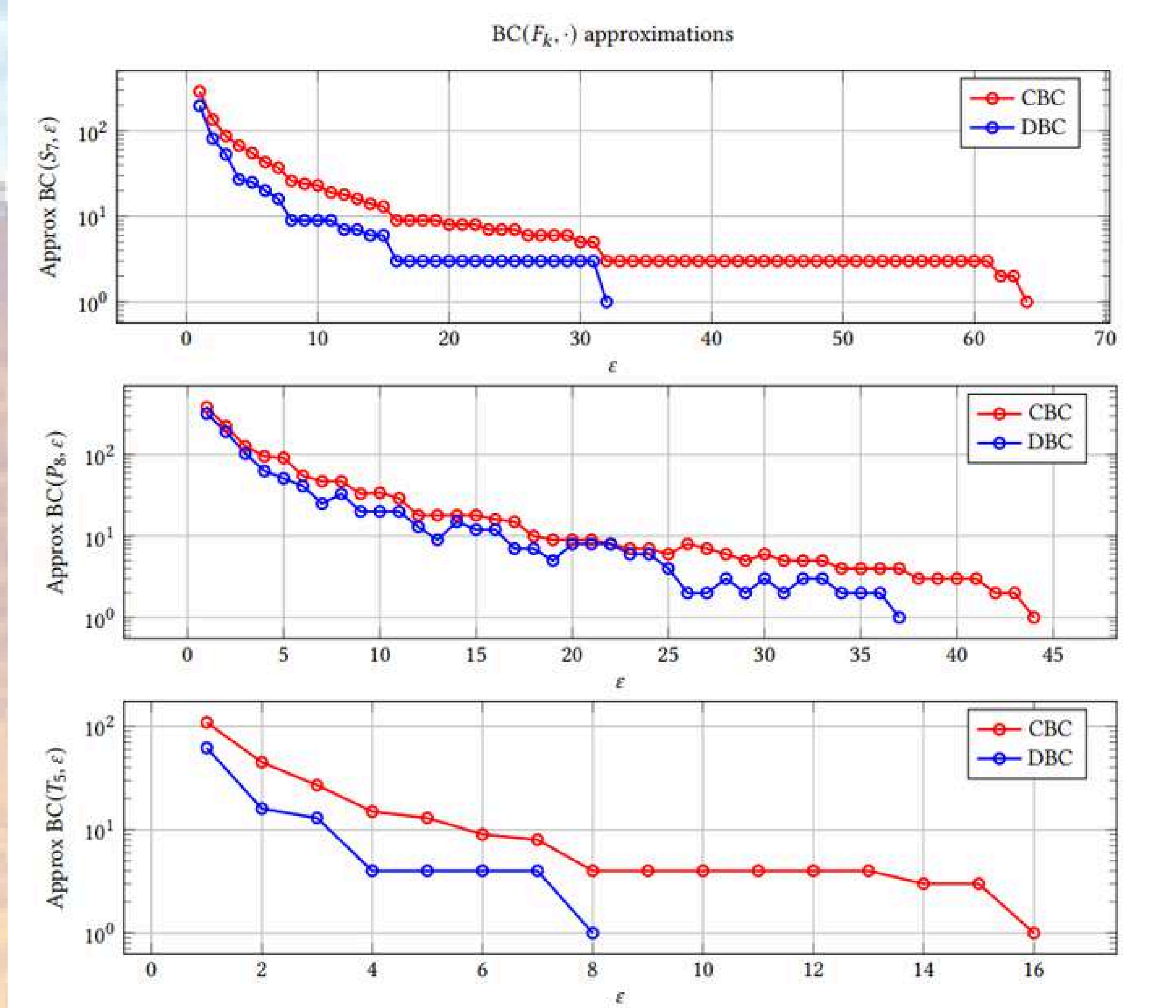
COROLLARY 8.11.2. *If we take* $f\left(\overline{F_n}, \varepsilon\right)$ *as an approximation for box-counting optimum values, Schroeder's conjecture ("in case of exact self-similarity,* D_H *and* D_{box} *are equal") is true for fractals in* $\mathcal{F}$.

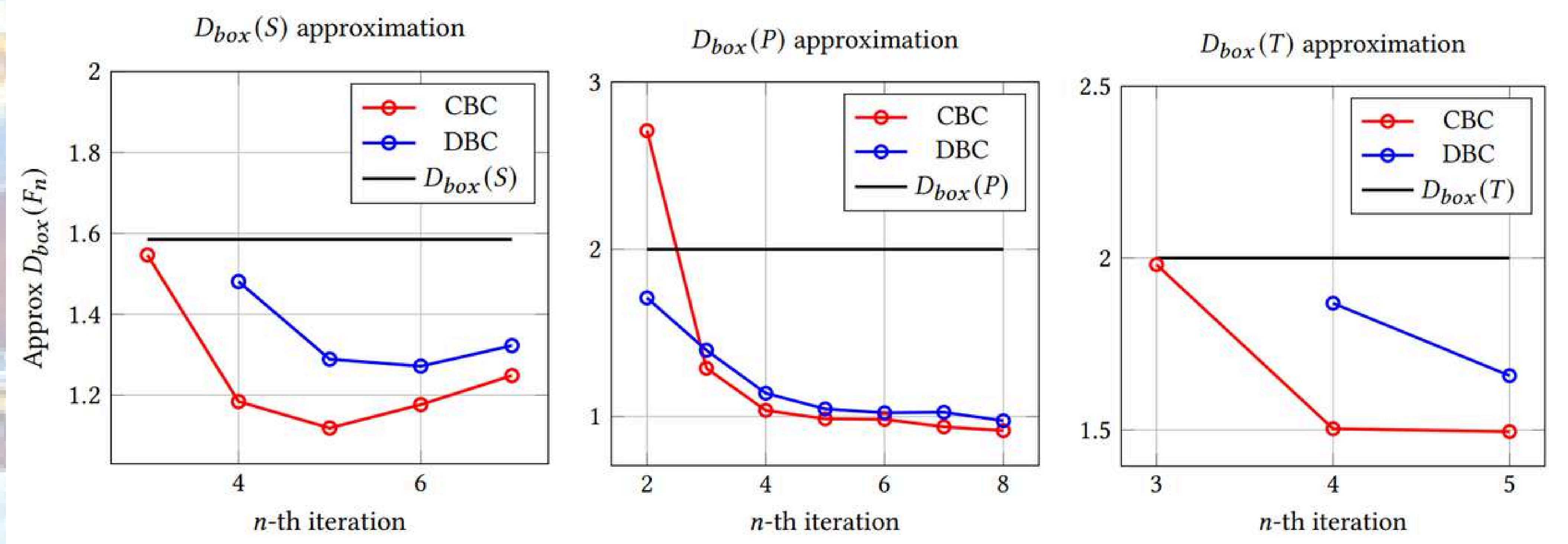
The weaker version of this corollary (but which doesn't require the validity of the approximation) is:

$$\forall F \in \mathcal{F},\ K_l \times D_H(F) \leq D_{box}(F) \leq K_u \times D_H(F)$$

YZU - CSDBCAFGDC : 9. Testing

Box-counting fractal dimension computation Java framework results





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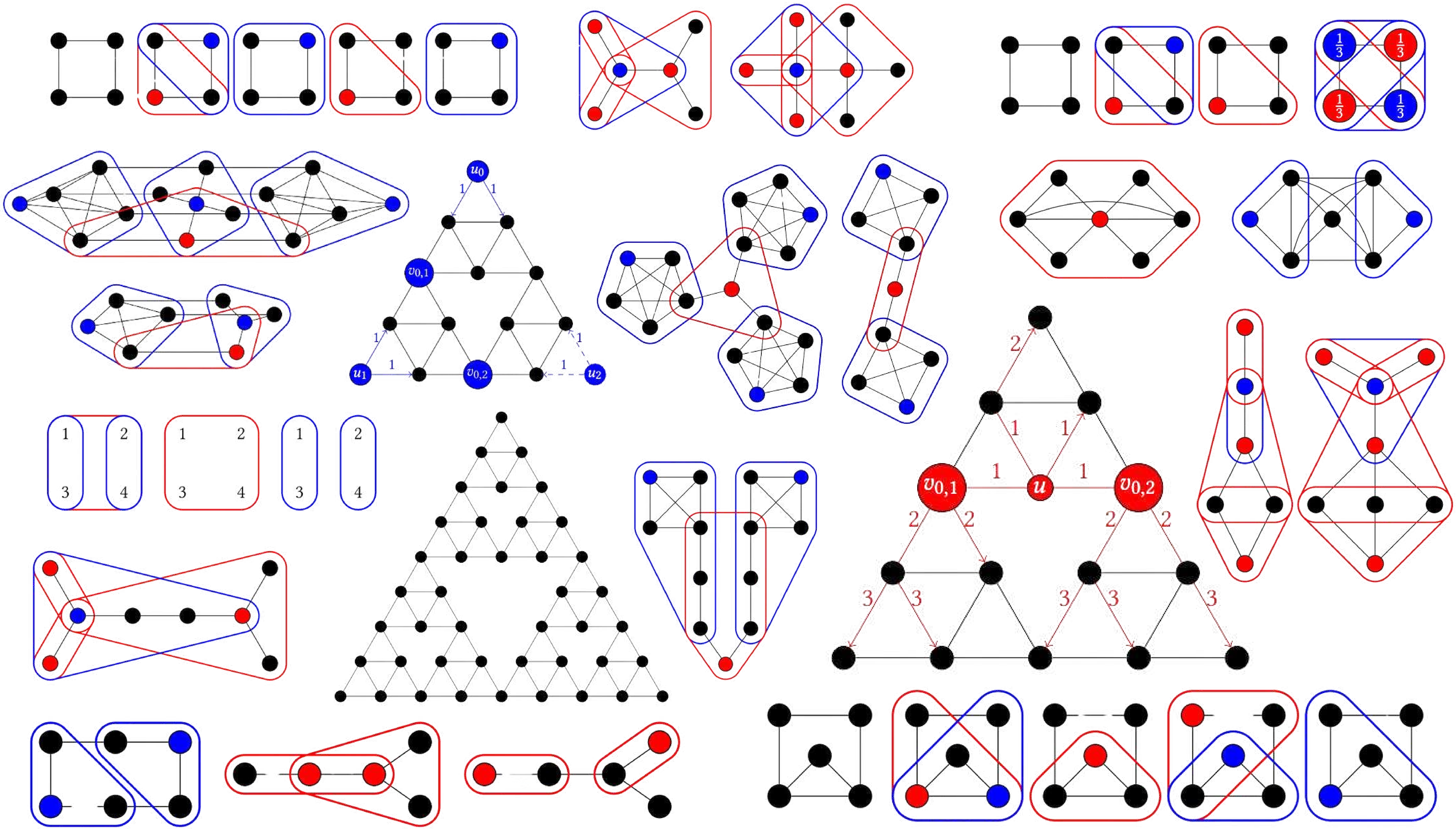
Thank you for your attention

ANY QUESTIONS ?



$\frac{1}{3}$ $\frac{1}{3}$ $\frac{1}{3}$ $\frac{1}{3}$
u_0
$v_{0,1}$
u_1
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u_2
1 2 3 4
1 2 3 4
1 3
2 4
$v_{0,1}$
u
$v_{0,2}$